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Ideal solidification of a liquid-metal boundary layer flow over a conveying substrate

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The ideal solidification problem of the two-dimensional boundary layer flow of a superheated liquid-metal binary alloy over a conveying substrate is examined. Analytical results for the velocities, the solute concentration and the temperature field are found in the asymptotic limits of small Prandtl numbers and large Schmidt numbers. The growth of the solidifying front is shown to follow the square-root law.

1. Introduction

In view of today's scientific work, which is almost exclusively based on numerical calculations, the need of approximate analytical solutions is greater than ever, especially as independent checks on the validity of numerical results. Exact and approximate analytical solutions describing the complex interactions between heat and fluid flow accompanied by phase change are exceptionally rare. The primary difficulties are due to the nonlinearity of the Navier–Stokes equations and the unknown position of the solidifying interface. This rules out the use of the superposition principle to build up complicated solutions from simple ones. This severely restricts our scope for problems having particular simple geometries for which a similarity solution exists. Consequently, analytical solutions to coupled solidification problems are most likely to be found in cases where a certain similarity variable can be distinguished.

One of the few coupled solidification problems that admit a similarity solution is the two-dimensional viscous stagnation flow against a plane solidification front for which Bian & Rangel (1996) propose a quasi-steady solution. Another example is presented by Löfgren & Åkerstredt (2000) where the stagnation-point flow against a conveyed solidification front, appearing in connection with the direct strip casting process, is considered. (The direct strip casting process was formerly called horizontal belt strip casting in earlier papers by the author.)

Another solidification problem of central importance in the industrial process of continuous strip casting is the boundary layer flow of a solidifying liquid metal over a heat-extracting moving substrate. The liquid metal is here directed onto a conveyor belt or a spinning chill-wheel by a nozzle. At the heat extracting moving boundary a growing solidified shell of steady-state shape emerges that is continuously withdrawn in the casting direction. The actual shape of the solidifying interface is controlled primarily by the heat transfer and the velocity of the moving substrate. Analytical treatment of this case was first considered by Katgerman (1980) and more recently by Carpenter & Steen (1997) and Löfgren & Åkerstedt (2000, 2001).

In the present paper we wish to draw attention to a certain similarity solution concerning the ideal solidification of a superheated liquid-metal boundary layer flow over a horizontally moving substrate. In this solution we simultaneously solve the viscous, solute and thermal fields together with the growth of the solid phase. For comparison and for later reference we here briefly sketch the corresponding Stefan problem without superheat for which the heat and the fluid flow are completely uncoupled.

Consider the two-dimensional problem of a liquid metal flow of zero superheat over a horizontally moving substrate at constant temperature T_0 and velocity V. Then in Cartesian coordinates the heat equation is given by

$$V\frac{\partial T_s}{\partial x} = \alpha_s\left\{\frac{\partial^2 T_s}{\partial x^2} + \frac{\partial^2 T_s}{\partial y^2}\right\}, \tag{1}$$

where x and y are the horizontal and vertical coordinates and α_s is the thermal diffusivity of the solidified metal.

The boundary conditions are

$$T_s(x,0) = T_0 \quad \text{and} \quad T_s(x,s(x)) = T_I, \tag{2}$$

where T_I is the temperature of the solidifying interface, and the energy balance at the solidifying interface $y = s(x)$

$$\kappa_s \hat{\boldsymbol{n}} \cdot \nabla T_s = V\Delta h_f \frac{\mathrm{d}s}{\mathrm{d}x}. \tag{3}$$

Here $\hat{\boldsymbol{n}}$ is the unit normal vector of the interface, directed into the melt, κ_s the thermal conductivity of solid metal, and Δh_f the latent heat of fusion per unit volume.

Introducing the similarity variable $\eta = y/s(x)$, a self-consistent solution is found in the limit $x^{1/2} \gg \Phi_0(\alpha_s/V)^{1/2}$, that is

$$T_s(\eta) = T_0 + \frac{T_I - T_0}{\text{erf}\ \Phi_0}\text{erf}\ (\Phi_0\eta), \tag{4}$$

where the growth of the solidification front is

$$s(x) = 2\Phi_0(\alpha_s x/V)^{1/2}. \tag{5}$$

The solidification constant Φ_0 is then defined by

$$\sqrt{\pi}\Phi_0 \exp \Phi_0^2 \,\text{erf}\ \Phi_0 = St, \tag{6}$$

found by substitution of (4) and (5) into condition (3), where $St = c_s(T_I - T_0)/\Delta h_f$ is the Stefan number and $c_s = \kappa_s/\alpha_s$ the specific heat of the solid metal.

It is in fact in the introduction of the similarity variable η where the success of the boundary layer analysis of the present paper rests. A natural question to be raised at this point is how will the growth of the solidification front be influenced if a superheated boundary layer flow sweeps the solidifying interface?

This paper is organized as follows: In § 2 we formulate the problem for the ideal solidification of the liquid-metal boundary layer flow over a moving substrate, concerning the velocity, solute and heat, and propose a square-root growth of the solidifying front. In § 3 we present an asymptotic solution for the viscous boundary layer flow in the limit of small Prandtl numbers. In § 4 we present an asymptotic solution for the solute boundary layer in the limit of large Schmidt numbers where the Prandtl number is proposed to be small but fixed. An asymptotic solution of the thermal boundary layer problem in the limit of small Prandtl numbers, and the thermal conduction through the solid phase is presented in § 5 and § 6, respectively. In § 7 the solidification constant is determined that thereby proves the proposed square root growth to be consistent.

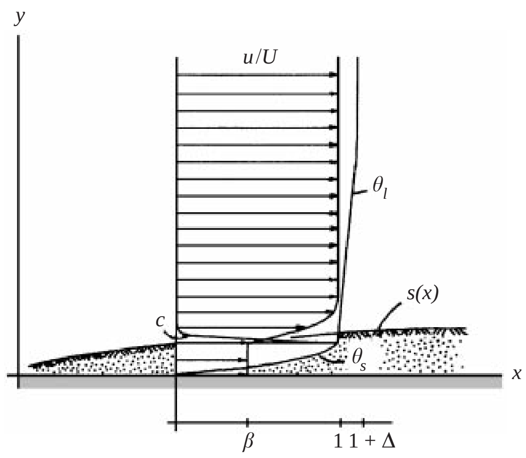

FIGURE 1. Boundary layer flow over an ideal solidification front. θ and c are the dimensionless temperature and solute concentration, respectively.

2. Problem formulation

In continuous strip casting, superheated liquid metal is fed through a nozzle onto a moving substrate where heat is extracted and from which the resulting product is collected. Some typical casting techniques are the direct strip casting process (DSC) and spin casting, i.e. chill-block, planar-flow casting (PFC) and single roll casting. For recent engineering advances in the DSC technique we refer to Nyström, Reichelt & Dubke (2000), while Steen & Karcher (1997) review the spin casting techniques. In contrast to the unidirectional solidification of binary alloys where the crystals are grown by pulling the crystals at a steady velocity through a furnace which maintains a controlled temperature profile in the melt and crystal (Kurz & Fisher 1992), the crystals in continuous strip casting grow under varying conditions. These castings are characterized by an approximately parallel film flow, excluding the narrow region at the feeding point. At some distance downstream, the superheated melt reaches fusion temperature and a time-independent solidification front emerges. Except for a small region close to the feeding point, the heat and fluid flow are only weakly coupled and interact primarily through the shape of the solidification front. The distributions of heat, solute and velocity are here homogenous apart from thin boundary layers close to the solidification front. It is the interactions of these boundary layers with the solidifying interface that we will examine, see figure 1.

To approximate the real boundary layer and solidification problem we make the following assumptions:

(i) Constant-temperature heat sink in perfect contact with the solidified metal. This would be a reasonable assumption for large Newtonian heat transfer coefficients or in the limit of large distances in the casting direction, see Löfgren & Åkerstedt (2000).

(ii) All material properties are constant.

(iii) The metal solidifies with a constant solute concentration kC^* (i.e. neglecting the initial transient), where C^* is the *a priori* unknown liquid interface concentration and k is the equilibrium partition ratio. The concentration C^* is determined as a part of the solution of the solute boundary layer problem. The far-field solute concentration is C_0.

(iv) The interface is in thermodynamic equilibrium, meaning that the temperature across the interface is continuous with the constant interface temperature $T_I = T_f + mC^*$. T_f is the fusion temperature of the pure metal and m is the liquidus slope (Kurz & Fisher 1992).

(v) No pressure gradient in the casting direction. This is valid for film flows with a free-stream velocity $U \gg (gh)^{1/2}$, where g is the acceleration due to gravity and h is the height of the free surface.

(vi) The ratio between the velocities of the moving substrate and the free stream $V/U = \beta$ is $Pr \ll \beta \ll Pr^{-1}$, where $Pr = \nu/\alpha_l$ is the Prandtl number. This is a reasonable assumption for liquid metals where Pr is a small parameter, typically between 0.1 and 0.01. (This restriction is only necessary when seeking asymptotic solutions in the limit of small Prandtl numbers, where we consider the ratio β to be of order unity.)

The mathematical problem is formulated in a Cartesian coordinate system (x, y), with the x-coordinate in the casting direction and the y-coordinate normal to the heat-extracting boundary; u and v are the corresponding velocity components. The viscous boundary layer approximation is then given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{8}$$

where ν is the kinematic viscosity.

Introducing the stream function $\psi(x, y)$, such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, the viscous boundary layer equation becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3}. \tag{9}$$

The solute and the thermal boundary layer equations are defined accordingly:

$$\frac{\partial \psi}{\partial y}\frac{\partial C}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2}, \tag{10}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial T_l}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T_l}{\partial y} = \alpha_l\frac{\partial^2 T_l}{\partial y^2}, \tag{11}$$

where D and α_l are the solute and thermal diffusivities of the liquid metal.

The heat equation in the solid phase is

$$V\frac{\partial T_s}{\partial x} = \alpha_s\frac{\partial^2 T_s}{\partial y^2}. \tag{12}$$

The boundary conditions are defined by

$$\left.\begin{array}{llll} \dfrac{\partial \psi}{\partial y} = V, & \psi = Vs & \text{at} & y = s(x), \\ \dfrac{\partial \psi}{\partial y} \to U & & \text{as} & y \to \infty, \end{array}\right\} \tag{13}$$

$$\left.\begin{array}{llll} C = C^*, & D\nabla C \cdot \hat{\boldsymbol{n}} = V(1-k)C^*\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}} & \text{at} & y = s(x), \\ C \to C_0. & & \text{as} & y \to \infty, \end{array}\right\} \tag{14}$$

$$\left.\begin{aligned} T_s &= T_0, && \text{at} \quad y = 0, \\ T_l &= T_s = T_I, && \text{at} \quad y = s(x), \\ T_l &\to T_I + \Delta T && \text{as} \quad y \to \infty, \end{aligned}\right\} \tag{15}$$

and the energy balance

$$\hat{\boldsymbol{n}} \cdot (\kappa_s \nabla T_s - \kappa_l \nabla T_l) = \Delta h_f \frac{\mathrm{D}s}{\mathrm{D}t} \quad \text{at} \quad y = s(x), \tag{16}$$

where ΔT is the superheat and κ_l is the thermal conductivity of the liquid metal.

In the following analysis we will assume the growth of the solidification front to have a square-root behaviour analogous to that found in the solution of the Stefan problem described in the introduction. This means that we expect the growth to be of the form

$$s(x) = 2\Phi \left(\frac{\alpha_s x}{V}\right)^{1/2} \quad \text{for} \quad x^{1/2} \gg \Phi(\alpha_s/V)^{1/2}, \tag{17}$$

where the solidification constant Φ is a free parameter left to be specified by the solution of the complete problem.

3. The solution of the viscous boundary layer problem

Introduce the new similarity variable

$$\eta = \frac{y}{2\Phi}\left(\frac{V}{\alpha_s x}\right)^{1/2}, \tag{18}$$

with $\eta = 1$ at the interface. Note that this variable bears a striking resemblance to the similarity variable used by Takeshita & Shingu (1983) in their study of the boundary layer flow over a conveying substrate without solidification. The corresponding stream function is of the form

$$\psi(x, y) = 2\Phi U \left(\frac{\alpha_s x}{V}\right)^{1/2} f(\eta), \tag{19}$$

where f is a dimensionless function. Substitution of (19) into (9) gives

$$Prf''' + 2\frac{A^2}{\beta} f f'' = 0, \quad A^2 = \alpha_{sl}\Phi^2; \quad \alpha_{sl} = \alpha_s/\alpha_l \tag{20}$$

where the primes denotes differentiation with respect to η. The boundary conditions to be imposed on f, found by substituting (19) into conditions (13), are

$$f = f' = \beta \quad \text{at} \quad \eta = 1 \quad \text{and} \quad f' \to 1 \quad \text{as} \quad n \to \infty. \tag{21}$$

In finding a solution to (20) with conditions (21) we make use of the fact that the Prandtl number is a small parameter in metallic systems, typically between 0.1 and 0.01, while considering A and β to be of $O(1)$. Let us therefore seek an approximate solution in the limit of small Prandtl numbers. In this limit, f exhibit a boundary layer structure close to the interface $\eta = 1$. Hence, the growth of the viscous boundary layer is much weaker than the growth of the solidification front.

Utilizing the method of matched asymptotic expansions, we assume an outer solution of the form

$$f_{out}(\eta; Pr) \sim \delta_1(Pr) f_1(\eta) + \delta_2(Pr) f_2(\eta) + \cdots \quad \text{as} \quad Pr \to 0 \quad \text{with} \quad \eta > 0 \text{ fixed}. \tag{22}$$

The $\{\delta_n\}$ is an asymptotic sequence such that the f_n are all of order unity in the outer region. By substituting expansion (22) into the full problem and taking the limit $Pr \to 0$, we obtain without loss of generality

$$\delta_1(Pr) = 1, \tag{23}$$

and the first-order problem

$$f_l'' = 0 \quad \text{where} \quad f_1' \to 1 \quad \text{as} \quad \eta \to \infty. \tag{24}$$

The solution to the first-order approximation is

$$f_1(\eta) = \eta + a, \tag{25}$$

where a is a constant left to be specified in the matching process with the inner solution.

An obvious length scale of the inner problem is of course the boundary layer thickness. Let this thickness be of order $\varepsilon(Pr)$, where ε is a function that vanishes as its argument tends to zero. The appropriate magnified coordinate is given by $\zeta = (\eta - 1)/\varepsilon(Pr)$, where $\varepsilon(Pr)$ is still to be determined.

Assume an inner expansion, valid within the boundary layer, of the form

$$f_{in}(\eta; Pr) \sim \Delta_1(Pr)F_1(\zeta) + \Delta_2(Pr)F_2(\zeta) + \cdots \quad \text{as} \quad Pr \to 0 \quad \text{with} \quad \zeta > 0 \text{ fixed.} \tag{26}$$

Here $\{\Delta_n\}$ is an asymptotic sequence such that all F_n are of order unity in the boundary layer, where $\zeta = O(1)$. By substituting expansion (26) into the full problem and taking the limit $Pr \to 0$, we obtain without loss of generality

$$\Delta_1(Pr) = 1 \quad \text{and} \quad \Delta_2(Pr) = \varepsilon(Pr) = \frac{\beta}{2A^2}Pr, \tag{27}$$

and the sequence of differential equations

$$F_1^{(3)} + F_1F_1^{(2)} = 0, \quad F_1(0) = \beta, F_1^{(1)}(0) = 0, \tag{28}$$

$$F_2^{(3)} + F_1F_2^{(2)} = 0, \quad F_2(0) = 0, F_2^{(1)}(0) = \beta, \tag{29}$$

where the index between the brackets represent differentiation with respect to ζ.

In the solution of the first-order problem we notice that $\psi > 0$ and bounded within the boundary layer, cf. (19). Consequently, $f_{in} \sim F_1$ must be > 0 and bounded throughout the boundary layer. Then by rewriting the first-order differential-equation into the differential-integral form

$$F_1^{(1)} = F_1^{(2)}(0) \int_0^\zeta \exp\left(\int_0^\sigma F_1 \mathrm{d}s\right) \mathrm{d}\sigma$$

we conclude that $F_1^{(2)}(0)$ must be zero. Hence, the solutions to equations (28) and (29) are given by

$$F_1(\zeta) = \beta, \quad F_2(\zeta) = \beta(1+b)\zeta + b(\mathrm{e}^{-\beta\zeta} - 1), \tag{30}$$

where b is a constant to be determined in the matching process.

Let us rewrite the outer solution (25) using the inner variable

$$f_{out} = \varepsilon\zeta + 1 + a, \tag{31}$$

and match order by order with the inner solution

$$f_{in} = \beta + \beta(1+b)\varepsilon\zeta + \varepsilon b(\mathrm{e}^{-\beta\zeta} - 1), \tag{32}$$

in the limit $\varepsilon\zeta \to 0$ and $\zeta \to \infty$. This gives

$$a = (\beta - 1)\left(1 + \frac{\varepsilon}{\beta}\right) \quad \text{and} \quad b = \frac{1-\beta}{\beta}, \tag{33}$$

where we let the second-order correction to the outer solution appear in the constant a.

Finally, a uniformly valid composite expansion is determined, i.e.

$$f(\eta; Pr) \sim \eta - 1 + \beta + Pr\frac{\beta - 1}{2A^2}\left(1 - \exp\left(-\frac{2A^2}{Pr}(\eta - 1)\right)\right) + O(Pr^2). \tag{34}$$

Using the composite expansion (34) the dimensionless horizontal and vertical velocity components are given by

$$\frac{u}{U} = \frac{1}{U}\frac{\partial\psi}{\partial y} = f' \sim 1 + (\beta - 1)\exp\left(-\frac{2A^2}{Pr}(\eta - 1)\right) + O(Pr) \quad \text{as} \quad Pr \to 0 \tag{35}$$

and

$$\begin{aligned}\frac{v}{U} &= -\frac{1}{U}\frac{\partial\psi}{\partial x} \sim \Phi\left(\frac{\alpha_s}{Vx}\right)^{1/2}\{\eta f' - f\} \\ &= \Phi\left(\frac{\alpha_s}{Vx}\right)^{1/2}(1-\beta)\left\{1 - \eta\exp\left(-\frac{2A^2}{Pr}(\eta - 1)\right) + O(Pr)\right\} \quad \text{as} \quad Pr \to 0.\end{aligned} \tag{36}$$

In the forthcoming analysis we will need certain expansions of $\int f\mathrm{d}\eta$, stated here for future reference. The Taylor expansion at $\eta = 1$ (Pr fixed), using expression (34), is

$$\int_1^\eta f\mathrm{d}\eta = \beta(\eta - 1) + \tfrac{1}{2}(\eta - 1)^2 + \frac{1}{3Pr}A^2(1-\beta)(\eta - 1)^3 + \text{HOT}. \tag{37}$$

The asymptotic expansion, using expression (34), is

$$\int_1^\eta f\mathrm{d}\eta \sim \tfrac{1}{2}(\eta - 1)^2 + \beta(\eta - 1) + Pr\frac{\beta - 1}{2A^2}(\eta - 1) + O(Pr^2) \quad \text{as} \quad Pr \to 0. \tag{38}$$

4. The solution of the solute boundary layer problem

Introduce the similarity variable (18) and the dimensionless solute concentration

$$c = \frac{C - C_0}{C^* - C_0}, \tag{39}$$

assuming $c = c(\eta)$. Substitution of expressions (19) and (39) into equation (10) yields

$$\frac{Pr}{Sc}c'' + 2\frac{A^2}{\beta}fc' = 0, \tag{40}$$

where $Sc = \nu/D$ is the Schmidt number and the primes denotes differentiation with respect to η. The boundary conditions to be imposed on c, found by substituting (39) into conditions (14), are

$$c = 1 \quad \text{and} \quad c' = -2A^2\frac{(1-k)C^*}{C^* - C_0}\frac{Sc}{Pr} \quad \text{at} \quad \eta = 1 \quad \text{and} \quad c \to 0 \quad \text{as} \quad \eta \to \infty, \tag{41}$$

provided $x^{1/2} \gg \Phi(\alpha_s/V)^{1/2}$ so that $\hat{\boldsymbol{n}} \sim \hat{\boldsymbol{y}}$. The straightforward solution is

$$c(\eta; Pr, Sc) = 1 - \frac{I(\eta; Pr, Sc)}{I(\infty; Pr, Sc)}, \quad \text{where} \quad I(\eta; Pr, Sc) = \int_1^{\eta} \exp\left(-2\frac{A^2}{\beta}\frac{Sc}{Pr}\int_1^{\xi} f\,\mathrm{d}\sigma\right)\mathrm{d}\xi, \tag{42}$$

with the *a priori* unknown liquid interface concentration C^* given by

$$C^* = \frac{C_0}{1 - 2A^2(1-k)(Sc/Pr)I(\infty; Pr, Sc)}. \tag{43}$$

In real alloy systems the Schmidt number is large, typically between 10 and 100, so in order to find an asymptotic expansion of the integral $I(\eta; Pr, Sc)$ we may take the limit $Sc \to \infty$ with Pr small but fixed. In this limit the integrand rapidly goes to zero even for η close to 1. This allows us to use Laplace method i.e. expanding the exponent in its Taylor series (37), giving

$$\begin{aligned} I(\eta; Pr, Sc) &\sim \int_1^{\eta} \left(1 - \frac{A^2}{\beta}\frac{Sc}{Pr}(\xi-1)^2 + O\left(\frac{Sc}{Pr^2}(\xi-1)^3\right)\right)\exp\left(-2A^2\frac{Sc}{Pr}(\xi-1)\right)\mathrm{d}\xi \\ &= \frac{Pr}{Sc}\frac{1-\exp\left(-2A^2\dfrac{Sc}{Pr}(\eta-1)\right)}{2A^2} + \left(\frac{Pr}{Sc}\right)^2\frac{F(\eta; Sc/Pr)}{8A^4\beta} \\ &\quad + O\left(\frac{Pr^2}{Sc^3}\right) \quad \text{as} \quad Sc \to \infty, \end{aligned} \tag{44}$$

where

$$F(\eta; Sc/Pr) = (\varphi^2 + 2\varphi + 2)\mathrm{e}^{-\varphi} - 2 = O(1), \quad \varphi = 2A^2\frac{Sc}{Pr}(\eta - 1). \tag{45}$$

Thus the solute concentration is then described by

$$c(\eta; Sc/Pr) \sim \mathrm{e}^{-\varphi} - \frac{Pr}{Sc}\frac{\varphi(\varphi+2)\mathrm{e}^{-\varphi}}{4A^2\beta} + O\left(\frac{Pr}{Sc^2}\right) \quad \text{as} \quad Sc \to \infty. \tag{46}$$

The influence of the viscous boundary layer flow is of $O(Pr/Sc^2)$ and therefore negligible in this limit.

Bird, Stewart & Lightfoot (1960) conduct a similar analysis of the concentration field in a gaseous boundary layer flow over a volatile solid plate that sublimes, under steady conditions, into the unbounded stream. Physically this case is quite different from ours; however the mathematical problem is largely the same.

Furthermore, the asymptotic expansion for the liquid interface concentration C^* is given by

$$C^* = C_0\left(\frac{1}{k} - \frac{1-k}{2\beta A^2k^2}\frac{Pr}{Sc} + O\left(\frac{Pr}{Sc^2}\right)\right) \quad \text{as} \quad Sc \to \infty. \tag{47}$$

This expansion should be compared to the liquid interface concentration C_0/k that appears for the well-known steady-state diffusion field ahead of a planar interface moving at constant velocity (Kurz & Fisher 1992). We thereby conclude that the ideally solidifying liquid-metal boundary layer flow will solidify at a lower solute concentration than for the steady-state solidification of a planar interface moving at constant velocity.

5. The solution of the thermal boundary layer problem

Introduce the similarity variable (18) and the dimensionless liquid temperature

$$\theta_l = \frac{T_l - T_0}{T_I - T_0}, \tag{48}$$

assuming $\theta_l = \theta_l(\eta)$. Substitution of (19) and (48) into (11) yields

$$\theta_l'' + 2\frac{A^2}{\beta} f \theta_l' = 0, \tag{49}$$

where the primes denote differentiation with respect to η. The boundary conditions to be imposed on θ_l, found by substituting (48) into conditions (15), are

$$\theta_l = 1 \quad \text{at} \quad \eta = 1 \quad \text{and} \quad \theta_l \to 1 + \Delta \quad \text{as} \quad \eta \to \infty, \tag{50}$$

where $\Delta = \Delta T/(T_I - T_0)$ is the dimensionless superheat. The straightforward solution is

$$\theta_l(\eta; Pr) = 1 + \frac{\Delta}{J(\infty; Pr)} J(\eta; Pr), \quad \text{where} \quad J(\eta; Pr) = \int_1^\eta \exp\left(-2\frac{A^2}{\beta}\int_1^\xi f \,\mathrm{d}\sigma\right) \mathrm{d}\xi. \tag{51}$$

The asymptotic expansion of the integral J in the limit of small Pr is found using (38), hence

$$\begin{aligned} J(\eta; Pr) &\sim \int_1^\eta \exp\left(-\frac{A^2}{\beta}\left((\xi-1)^2 + 2\beta(\xi-1) + Pr\frac{(\beta-1)}{A^2}(\xi-1)\right)\right)\mathrm{d}\xi \\ &= \frac{(\pi\beta)^{1/2}}{2A}\exp\left(A\beta^{1/2} + Pr\frac{(\beta-1)}{2A\beta^{1/2}}\right)^2 \left\{\operatorname{erf}\left(\frac{A}{\beta^{1/2}}(\eta - 1 + \beta) + Pr\frac{(\beta-1)}{2A\beta^{1/2}}\right)\right. \\ &\quad \left. - \operatorname{erf}\left(A\beta^{1/2} + Pr\frac{(\beta-1)}{2A\beta^{1/2}}\right)\right\} \quad \text{as} \quad Pr \to 0. \end{aligned} \tag{52}$$

Thus the temperature field becomes

$$\theta_l(\eta; Pr) \sim 1 + \Delta\frac{\operatorname{erf}\left(\frac{A}{\beta^{1/2}}(\eta - 1 + \beta) + Pr\frac{(\beta-1)}{2A\beta^{1/2}}\right) - \operatorname{erf}\left(A\beta^{1/2} + Pr\frac{(\beta-1)}{2A\beta^{1/2}}\right)}{\operatorname{erfc}\left(A\beta^{1/2} + Pr\frac{(\beta-1)}{2A\beta^{1/2}}\right)} \quad \text{as} \quad Pr \to 0. \tag{53}$$

It can readily be seen in expression (53) that the influence of the viscous boundary layer flow, the term of $O(Pr)$, is weak and therefore negligible in a first approximation.

6. The solution of the thermal problem in the solidified phase

Introduce the similarity variable (18) and the dimensionless temperature

$$\theta_s = \frac{T_s - T_0}{T_I - T_0}, \tag{54}$$

assuming $\theta_s = \theta_s(\eta)$. Substitution of (19) and (54) into (12) yields

$$\theta_s'' + 2\Phi^2 \eta \theta_s' = 0, \tag{55}$$

where the primes denote differentiation with respect to η. The boundary conditions to be imposed on θ_s, found by substituting (54) into conditions (15), are

$$\theta_s = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \theta_s = 1 \quad \text{at} \quad \eta = 1. \tag{56}$$

The temperature distribution within the solid phase is

$$\theta_s(\eta) = \frac{\text{erf}(\Phi\eta)}{\text{erf}\,\Phi}, \tag{57}$$

which of course is analogous to expression (4) in the introduction.

7. Determination of the solidification constant Φ

The solidification constant Φ can now be determined through the use of the energy balance at the solidifying interface (16). Substituting the expressions (17), (48) and (54) into condition (16) yields

$$\theta_s'(1) - \kappa_{ls}\theta_l'(1; Pr) = 2\frac{\Phi^2}{St}, \quad \kappa_{ls} = \kappa_l/\kappa_s \tag{58}$$

provided $x^{1/2} \gg \Phi(\alpha_s/V)^{1/2}$ so that $\hat{\boldsymbol{n}} \sim \hat{\boldsymbol{y}}$. From expressions (53) and (57) we get

$$\theta_l'(1; Pr) \sim 2\Delta\left(\frac{\alpha_{sl}}{\pi\beta}\right)^{1/2} \frac{\Phi\exp\left(-\Phi^2\alpha_{sl}\beta\right)}{\text{erfc}(\Phi(\alpha_{sl}\beta)^{1/2})} + O(\Delta Pr) \quad \text{as} \quad Pr \to 0 \tag{59}$$

and

$$\theta_s'(1) = \frac{2}{\sqrt{\pi}}\frac{\Phi}{\text{erf}\,\Phi\exp\Phi^2}. \tag{60}$$

Hence, in a first approximation with an error of $O(\Delta Pr)$ we have

$$\sqrt{\pi}\Phi\,\text{erf}\,\Phi\exp\Phi^2 = St - \Delta St\kappa_{ls}\left(\frac{\alpha_{sl}}{\beta}\right)^{1/2}\frac{\text{erf}\,\Phi\exp\left(\Phi^2(1-\alpha_{sl}\beta)\right)}{\text{erfc}\left(\Phi(\alpha_{sl}\beta)^{1/2}\right)}. \tag{61}$$

It is interesting to note that equation (61) will give $\Phi > 0$ for every Δ. Hence the square root growth of the solidifying front is a direct consequence of the ideal cooling assumption.

In real casting situations the dimensionless superheat Δ is typically very much less than unity. So, in order to make a first step in determining the effect of superheated melt on the growth of the solidification front we may consider the limit $\Delta \to 0$ and assume the regular perturbation expansion

$$\Phi = \Phi_0 + \Delta\Phi_1 + \cdots. \tag{62}$$

Insertion of expansion (62) into (61) gives to the zeroth order the problem defined by (6). The first-order correction is then

$$\Phi_1 = -\kappa_{ls}\left(\frac{\alpha_{sl}}{\pi\beta}\right)^{1/2}\frac{St^2}{St + 2\Phi_0(\Phi_0 + St)}\frac{\exp\left(-\alpha_{sl}\beta\Phi_0^2\right)}{\text{erfc}\left(\Phi_0(\alpha_{sl}\beta)^{1/2}\right)}. \tag{63}$$

We hereby conclude that the influence of the viscous boundary layer, which is of $O(\Delta Pr)$, is negligible for the growth of the solidification front in the present application of continuous casting. The solidification problem is therefore almost purely a matter of decoupled heat and fluid flow.

To give a rough estimate of the solidification constants Φ_0 and Φ_1 we consider liquid steel with the physical properties $\Delta h_f \approx 1 \times 10^9\,\text{J m}^{-3}$, $c_s \approx 5 \times 10^6\,\text{J m}^{-3}\,\text{K}^{-1}$

and $\kappa_{ls} \approx \alpha_{sl} \approx 1$. Then, for a temperature difference of $T_I - T_0 \approx 1000\,\mathrm{K}$ between the solidifying interface and the heat sink the Stefan number is about 5. Using (6) and (63) the solidification constant $\Phi_0 \approx 1$ while Φ_1 takes values between -2 and -4 when $0.1 \leqslant \beta \leqslant 10$.

8. Summary

In this paper we analyse the ideal solidification of a liquid-metal boundary layer flow over a conveying substrate. This flow is of central importance in the industrial process of continuous strip casting. The liquid metal is here directed onto a conveyor belt or a spinning chill-wheel by a nozzle. At the heat-extracting moving boundary a growing solidified shell of steady-state shape emerges that is initially swept by a superheated boundary layer flow. It is the interaction between the viscous boundary layer and the solute and temperature boundary layers accompanied by the phase change that this paper concerns.

We show that the viscous, solute and thermal boundary layers can be described by a single differential equation through the use of the similarity variable $\eta = y/s(x)$, where s is the thickness of the solidified phase. Analytic solutions are found in the asymptotic limit of small Prandtl and large Schmidt numbers. These solutions are useful as independent checks on the validity of numerical calculations and necessary in the prediction of morphological instabilities of the solidifying interface, see the review paper by Davis & Schulze (1996).

The solidification front is shown to follow the square-root law and a solidification constant Φ is derived for general superheats. For weak superheats, such as appear in real strip casting applications, the influence of the viscous boundary layer on the growth is found to be negligible.

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